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Bounds on the a-invariant and reduction numbers of ideals

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Abstract

Let R be a d-dimensional standard graded ring over an Artinian local ring. Let $\mathfrak{M}$ be the unique maximal homogeneous ideal of R. Let $h^i(R)_n$ denote the length of the nth graded component of the local cohomology module $H^i_{\mathfrak{M}}(R)$. Define the Eisenbud–Goto invariant $EG(R)$ of R to be the number $\sum_{q=0}^{d-1}\binom{d-1}{q}h^q_{\mathfrak{M}}(R)_{1-q}$. We prove that the a-invariant $a(R)$ of the top local cohomology module $H^d_{\mathfrak{M}}(R)$ satisfies the inequality: $a(R) \leqslant e(R) - \ell(R_1) + (d-1)(\ell(R_0)-1) + EG(R)$. This bound is used to get upper bounds for the reduction number of an $\mathfrak{m}$-primary ideal I of a Cohen–Macaulay local ring $(R,\mathfrak{m})$, when the associated graded ring of I has depth at least $d-1$.

Keywords: a-invariant; Reduction number; Eisenbud–Goto invariant; Local cohomology

1. Introduction

Let $R = \bigoplus_{n=0}^{n=\infty} R_n$ be a d-dimensional standard graded ring over an Artinian local ring R_0. Let $\mathfrak{M}$ be the maximal homogeneous ideal of R. Let $H^i_{\mathfrak{M}}(R)$ denote the ith local cohomology module of R with respect to $\mathfrak{M}$. For a graded module M, we use $[M]_n$ or

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M_n to denote the nth graded component of M. The a-invariant of R, introduced in [3], is defined as

$$a(R) = \max\{n \mid [H^d_{\mathfrak{M}}(R)]_n \neq 0\}.$$

The objective of this paper is to give a bound for the a-invariant of R in terms of lengths of graded components of local cohomology modules and use it to get bounds for reduction numbers of ideals. Let $\ell(M)$ denote length of a module M. We set $\ell([H^q_{\mathfrak{M}}(R)]_{1-q}) = h^q(R)_{1-q}$ for all $q \geqslant 0$. To state our bound for the a-invariant we define the Eisenbud–Goto invariant $EG(R)$ of R to be the number

$$EG(R) = \sum_{q=0}^{d-1} \binom{d-1}{q} h^q(R)_{1-q}.$$

The main result of the paper is:

Theorem 1.1. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a d-dimensional standard graded algebra over an Artinian local ring R_0 with multiplicity $e(R)$. Then*

$$a(R) \leqslant e(R) - \ell(R_1) + (d-1)\big(\ell(R_0) - 1\big) + EG(R).$$

Eisenbud and Goto [2] showed that if R_0 is a field then

$$e(R) \geqslant 1 + \operatorname{codim}(R) - EG(R).$$

They showed that if equality holds in the above inequality then $R/H^0_{\mathfrak{M}}(R)$ has linear resolution. To state our bounds for reduction numbers we recall some basic concepts about reductions of ideals. Let $(R, \mathfrak{m})$ be a local ring. Let $J \subset I$ be ideals of R. The ideal J is called a reduction of I if there exists an $n \in \mathbb{N}$ such that $JI^n = I^{n+1}$ [7]. Among the reductions of I, the smallest ones with respect to inclusion are called minimal reductions of I. If $R/\mathfrak{m}$ is infinite then any minimal reduction of I is minimally generated by as many elements as the Krull dimension of the fiber cone $F(I) := \bigoplus_{n=0}^{n=\infty} I^n/\mathfrak{m}I^n$. The *reduction number,* $r_J(I)$, of I with respect to a minimal reduction J is the least integer n for which $JI^n = I^{n+1}$. When $R/\mathfrak{m}$ is infinite, the reduction number of I is defined as the minimum of the reduction numbers $r_J(I)$ where J varies over all the minimal reductions of I. Let $G(I) := \bigoplus_{n \geqslant 0} I^n/I^{n+1}$ be the associated graded ring of an ideal I. Let $\gamma(I)$ denote the depth of the irrelevant ideal G_+ of $G(I)$. If $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring, I is an $\mathfrak{m}$-primary ideal and $\gamma(I) \geqslant d-1$, then $r(I) = a(G(I)) + d$ [6]. The *Ratliff–Rush closure* of an ideal I, $\tilde{I}$, is the stable value of the sequence of the ideals $\{I^{n+1} : I^n\}$. We will obtain the following bounds for $r(I)$ as an application of the main theorem:

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen–Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal with $\gamma(I) \geqslant d-1$. Let J be any minimal reduction of I.*

(1) *Let* $d = 1$. *Then* $r(I) \leqslant e(I) - (\ell(I/(I \cap \tilde{I^2}) - 1) \leqslant e(I)$.
(2) *Let* $d = 2$. *Put* $X = \operatorname{Proj}(G(I))$. *Then* $r(I) \leqslant 1 + e(I) - \ell(I/I^2) + \ell(H^0(X, \mathcal{O}_X))$.
(3) *Let* $d \geqslant 3$. *Then* $r(I) \leqslant 1 + \ell(I^2/JI) + h^{d-1}(G(I))_{2-d}$.

We will show by an example that our bounds for the a-invariant and reduction number are sharp.

2. A bound on the a-invariant of standard graded algebras

In this section we prove our bound on the a-invariant of a standard graded algebra R over an Artinian local ring R_0.

Theorem 2.1. *Let* $R = \bigoplus_{n=0}^{n=\infty} R_n$ *be a* d*-dimensional standard graded algebra over an Artinian local ring* R_0. *Then*

$$a(R) \leqslant e(R) - \ell(R_1) + (d-1)\big(\ell(R_0) - 1\big) + EG(R). \tag{1}$$

Proof. We may assume without loss of generality that the residue field of R_0 is infinite. We prove the theorem by induction on d. Let $d = 0$. Then

$$e(R) = \ell(R_0) + \ell(R_1) + \cdots + \ell(R_m),$$

where $m = a_0(R)$. Thus

$$e(R) - \ell(R_1) - \ell(R_0) + 1 = 1 + \ell(R_2) + \cdots + \ell(R_m) \geqslant m.$$

Let R be Cohen–Macaulay and pick a degree one nonzerodivisor x to see that

$$\begin{aligned} a(R) &= a(R/xR) - 1 \\ &\leqslant e(R/xR) - \ell\big([R/xR]_1\big) + (d-2)\big(\ell(R_0) - 1\big) - 1 \\ &= e(R) - \ell(R_1) + \ell(R_0) + (d-2)\big(\ell(R_0) - 1\big) - 1 \\ &= e(R) - \ell(R_1) + (d-1)\big(\ell(R_0) - 1\big). \end{aligned}$$

Now let $d = 1$. If R is Cohen–Macaulay, we are done by the above argument. So let $\operatorname{depth}(R) = 0$. Then $S := R/H^0_{\mathfrak{M}}(R)$ is Cohen–Macaulay, $e(S) = e(R)$ and $a(R) = a(S)$. Hence

$$a(R) = a(S) \leqslant e(S) - \ell(S_1) = e(R) - \ell(R_1) + h^0(R)_1.$$

Suppose $d \geqslant 2$. Let $x \in R_1$ be a superficial element. We first prove that for a degree one superficial element in R,

$$EG(R/xR) \leqslant EG(R).$$

Since x is superficial of degree one,

$$H^i_{\mathfrak{M}}(0:_R x) = \begin{cases} (0:_R x) & \text{if } i = 0, \\ 0 & \text{if otherwise.} \end{cases}$$

Hence from the short exact sequence

$$0 \to (0:_R x) \to R \to \frac{R}{(0:_R x)} \to 0$$

we get $H^i_{\mathfrak{M}}(R/(0:_R x)) = H^i_{\mathfrak{M}}(R)$ for all $i \geqslant 1$. From the exact sequence

$$0 \to \frac{R}{(0:_R x)}(-1) \to R \to \frac{R}{xR} \to 0$$

we get the long exact sequence

$$\cdots \to \big[H^i_{\mathfrak{M}}(R)\big]_n \to \big[H^i_{\mathfrak{M}}(R/xR)\big]_n \to \big[H^{i+1}_{\mathfrak{M}}(R)\big]_{n-1} \to \big[H^{i+1}_{\mathfrak{M}}(R)\big]_n \to \cdots.$$

Hence for all $i \geqslant 0$,

$$h^i(R/xR)_n \leqslant h^i(R)_n + h^{i+1}(R)_{n-1}.$$

Hence

$$\begin{aligned} EG(R/xR) &= \sum_{q=0}^{d-2} \binom{d-2}{q} h^q(R/xR)_{1-q} \\ &\leqslant \sum_{q=0}^{d-2} \binom{d-2}{q} \big[h^q(R)_{1-q} + h^{q+1}(R)_{-q}\big] \\ &= \sum_{q=0}^{d-2} \binom{d-2}{q} h^q(R)_{1-q} + \sum_{q=1}^{d-1} \binom{d-2}{q-1} h^q(R)_{1-q} \\ &= \sum_{q=0}^{d-1} \binom{d-1}{q} h^q(R)_{1-q} \\ &= EG(R). \end{aligned}$$

Therefore

$$\begin{aligned} a(R) &\leqslant a(R/xR) - 1 \quad \text{by [9]} \\ &\leqslant e(R/xR) - \ell(R/xR)_1 + (d-2)\big(\ell\big([R/xR]_0\big) - 1\big) + EG(R) - 1 \\ &= e(R) - \ell(R_1) + \ell(R_0) - \ell\big((0:x)_{R_0}\big) + (d-2)\big(\ell(R_0) - 1\big) + EG(R) - 1 \end{aligned}$$

$$\leqslant e(R) - \ell(R_1) + (d-1)\big(\ell(R_0) - 1\big) + EG(R). \qquad \square$$

We now demonstrate that the bound in Theorem 2.1 is sharp.

Example 2.2. Let k be a field and x, y, a, b, c, d be indeterminates. Consider the ideal $I = (x^3, x^2y^4, xy^5, y^7)$ in the polynomial ring $S = k[x, y]$. Using Hilbert series we show that $F(I) \simeq k[a,b,c,d]/(bd, bc, b^2, c^3)$. Consider the ring homomorphism $\phi : R = k[a,b,c,d] \to F(I)$ defined by

$$\phi(a) = \overline{x^3}, \quad \phi(b) = \overline{x^2y^4}, \quad \phi(c) = \overline{xy^5}, \quad \text{and} \quad \phi(d) = \overline{y^7}.$$

Here the overbar indicates the image in the first graded component of $F(I)$. Let $L = \ker\phi$. The equations

$$\begin{aligned} &\big(x^2y^4\big)\big(y^7\big) = \big(xy^5\big)^2 y,\\ &\big(x^2y^4\big)\big(xy^5\big) = \big(x^3y^7\big)y^2,\\ &\big(x^2y^4\big)^2 = \big(x^3\big)\big(xy^5\big)y^3,\\ &\big(xy^5\big)^3 = x^3\big(y^7\big)^2 y, \end{aligned}$$

show that $N = (bd, bc, b^2, c^3) \subset L$. To show that $N = L$, we show that R/N and R/L have same Hilbert series. We denote the Hilbert series of a graded algebra G by $H(G,\lambda)$. By Propositions 2.3 and 2.6 of [5] we find that $\mu(I^n) = 3n + 1$ for all $n \geqslant 0$. Here μ denotes the minimum number of generators. This shows that

$$H\big(F(I), \lambda\big) = (1 + 2\lambda)/(1-\lambda)^2.$$

By the well known "divide and conquer strategy" for finding Hilbert series of quotients of polynomial rings by monomial ideals we get, $H(R/N, \lambda) = H(F(I), \lambda)$. Thus $F(I) \cong R/N$. Therefore $F(I)$ is a two-dimensional ring with depth one. Notice that $N = (b, c^3) \cap (c, d, b^2)$. Put $J = (b, c^3)$ and $K = (c, d, b^2)$. In order to get the desired information about local cohomology of $F(I)$, consider the exact sequence:

$$0 \to F(I) \to R/J \oplus R/K \to R/(J+K) \to 0.$$

Hence we get the following long exact sequence of local cohomology modules with respect to the maximal homogeneous ideal $\mathfrak{m} = (a, b, c, d)$:

$$0 \to H^1_{\mathfrak{m}}\big(F(I)\big) \to H^1_{\mathfrak{m}}(R/K) \to H^1_{\mathfrak{m}}\big(R/(J+K)\big) \to H^2_{\mathfrak{m}}\big(F(I)\big) \to H^2_{\mathfrak{m}}(R/J) \to 0.$$

We now show that $a(F(I)) = 0$ and $h^1(F(I))_0 = 1$. Since

$$R/J \simeq k[a,c,d]/\big(c^3\big), \quad R/(J+K) \simeq k[a] \quad \text{and} \quad R/K \simeq k[a,b]/\big(b^2\big),$$

by using that fact that $a(R/(f)) = a(R) + \deg(f)$ for a homogeneous regular element f of a graded algebra R, we conclude that $a(R/J) = 0$, $a(R/(J+K)) = -1$ and $a(R/K) = 0$. Thus $a(F(I)) = 0$. By [1, Theorem 4.4.3], we get

$$h^1\big(F(I)\big)_0 = h^1(R/K)_0 = P(R/K,0) - H(R/K,0) = 2 - 1 = 1.$$

Substituting these values in (1) we observe that equality holds. Therefore (1) is sharp.

3. Bounds on reduction numbers

In this section we will use the bound on the a-invariant obtained in the previous section to provide bounds on reduction numbers. By [9] and [6], we know that $r(I) = a(G(I)) + d$ where $(R,\mathfrak{m})$ is a Cohen–Macaulay local ring of dimension d and $\gamma(I) \geqslant d-1$. We will consider the cases where $d = 1$, $d = 2$ and $d \geqslant 3$ separately. In the next result we will need the formula: $[H^0_{G_+}(G(I))]_n = (I^n \cap \widetilde{I^{n+1}})/I^{n+1}$ for all $n \geqslant 0$ [4].

Proposition 3.1. *Let $(R,\mathfrak{m})$ be a one-dimensional Cohen–Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal. Then*

$$r(I) \leqslant e(I) - \big[\ell\big(I/\big(I \cap \tilde{I^2}\big)\big) - 1\big] \leqslant e(I).$$

Proof. Since $d = 1$,

$$\begin{aligned} a\big(G(I)\big) &\leqslant e(I) - \ell\big(I/I^2\big) + h^0(G)_1 \\ &= e(I) - \ell\big(I/I^2\big) + \ell\big(\big(I \cap \tilde{I^2}\big)/I^2\big) \\ &= e(I) - \ell\big(I/\big(I \cap \tilde{I^2}\big)\big). \end{aligned}$$

Hence $r(I) = a(G(I)) + 1 \leqslant 1 + e(I) - \ell(I/(I \cap \tilde{I^2})$. If $\ell(I/(I \cap \tilde{I^2}) = 0$ then $I \subseteq \tilde{I^2}$. But $\tilde{I^2} = I^{n+2} : I^n$ for large n. Hence $I^{n+1} = I^{n+2}$. This is a contradiction. Hence $\ell(I/(I \cap \tilde{I^2})) \geqslant 1$. Thus we obtain the classical bound $r(I) \leqslant e(I)$. □

Example 3.2. Let k be a field and t be an indeterminate. Put $R = k[[t^4, t^5, t^6, t^7]]$ and $I = (t^4, t^5, t^6)$. Let $\mathfrak{m}$ denote the unique maximal ideal of R. Let G denote the associated graded ring $G(I)$ of I. Then G is not Cohen–Macaulay since $t^7 I \subset I^2$. To find the associated Ratliff–Rush ideal of I notice that $I^2 = \mathfrak{m}^2$. Since $r(\mathfrak{m}) = 1$, the associated graded ring $G(\mathfrak{m})$ is Cohen–Macaulay by [8]. Therefore all powers of $\mathfrak{m}$ are Ratliff–Rush. Hence, $\tilde{I^2} = \widetilde{\mathfrak{m}^2} = \mathfrak{m}^2 = I^2$. Hence $(I \cap \tilde{I^2}/I^2) = 0$. Therefore $r(I) \leqslant 1 + e(I) - \ell(I/I^2) = 2$. It can be checked that $r_{(t^4)}(I) = 2$. Therefore the bound in the above result is sharp.

Proposition 3.3. *Let I be an $\mathfrak{m}$-primary ideal of a two dimensional Cohen–Macaulay local ring with $\gamma(I) \geqslant 1$. Let $X = \operatorname{Proj} G(I)$. Then*

$$r(I) \leqslant 1 + e(I) - \ell\big(I/I^2\big) + \ell\big(H^0(X, \mathcal{O}_X)\big).$$

Proof. Since $\gamma(I) \geqslant 1$,

$$r(I) \leqslant 1 + e(I) - \ell\big(I/I^2\big) + \ell(R/I) + h^1(G)_0.$$

By the exact sequence

$$0 \to H^0_{G_+}(G) \to G \to \bigoplus_{n \in \mathbb{Z}} H^0\big(X, \mathcal{O}_X(n)\big) \to H^1_{G_+}(G) \to 0$$

we get, by taking the 0th component of all the modules in the above exact sequence:

$$\ell\big(H^0(X, \mathcal{O}_X)\big) - \ell(R/I) = h^1(G)_0.$$

Putting this in the above bound for $r(I)$ we get the desired upper bound. □

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geqslant 3$ with $R/\mathfrak{m}$ infinite. Let I be an $\mathfrak{m}$-primary ideal with a minimal reduction J and $\gamma(I) \geqslant d - 1$. Then*

$$r(I) \leqslant 1 + \ell\big(I^2/JI\big) + h^{d-1}(G)_{2-d}.$$

Proof. Since $\gamma(I) \geqslant d - 1$, by Theorem 2.1

$$\begin{aligned}
r(I) &\leqslant e(I) - \ell\big(I/I^2\big) + (d-1)\big(\ell(R/I) - 1\big) + h^{d-1}(G)_{2-d} + d \\
&= \ell(R/J) - \ell\big(R/I^2\big) + \ell(R/I) + (d-1)\ell(R/I) - (d-1) + d + h^{d-1}(G)_{2-d} \\
&= \ell(R/J) - \ell\big(R/I^2\big) + \ell(J/JI) + 1 + h^{d-1}(G)_{2-d} \\
&= 1 + \ell\big(I^2/JI\big) + h^{d-1}(G)_{2-d}. \qquad \square
\end{aligned}$$

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References

[1] W. Bruns, J. Herzog, Cohen–Macaulay Rings, revised ed., in: Cambridge Stud. Adv. Math., vol. 39, 1998.
[2] D. Eisenbud, S. Goto, Linear free resolutions and minimal multiplicity, J. Algebra 88 (1984) 89–133.
[3] S. Goto, K. Watanabe, Graded rings I, J. Math. Soc. Japan 30 (1978) 179–213.
[4] W. Heinzer, B. Johnston, D. Lantz, K. Shah, Coefficient ideals in and blowups of a commutative Noetherian domain, J. Algebra 162 (1993) 355–391.
[5] C. Huneke, Complete ideals in two dimensional regular local rings, in: M. Hochster, C. Huneke, J.D. Sally (Eds.), Commutative Algebra, Springer-Verlag, 1989, pp. 325–338.
[6] T. Marley, The reduction number of an ideal and the local Cohomology of the associated graded ring, Proc. Amer. Math. Soc. 117 (1993) 335–341.
[7] D.G. Northcott, D. Rees, Reductions of ideals in local rings, Proc. Cambridge Philos. Soc. 50 (1954) 145–158.
[8] J.D. Sally, On the associated graded ring of a local Cohen–Macaulay ring, J. Math. Kyoto U. 17 (1977) 19–21.
[9] N.V. Trung, Reduction exponent and degree bound for defining equations of graded rings, Proc. Amer. Math. Soc. 101 (1987) 229–236.